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PROPAGATION OF ELECTROMAGNETIC WAVES ALONG CORRUGATED LINES

by

Georges G. Weill

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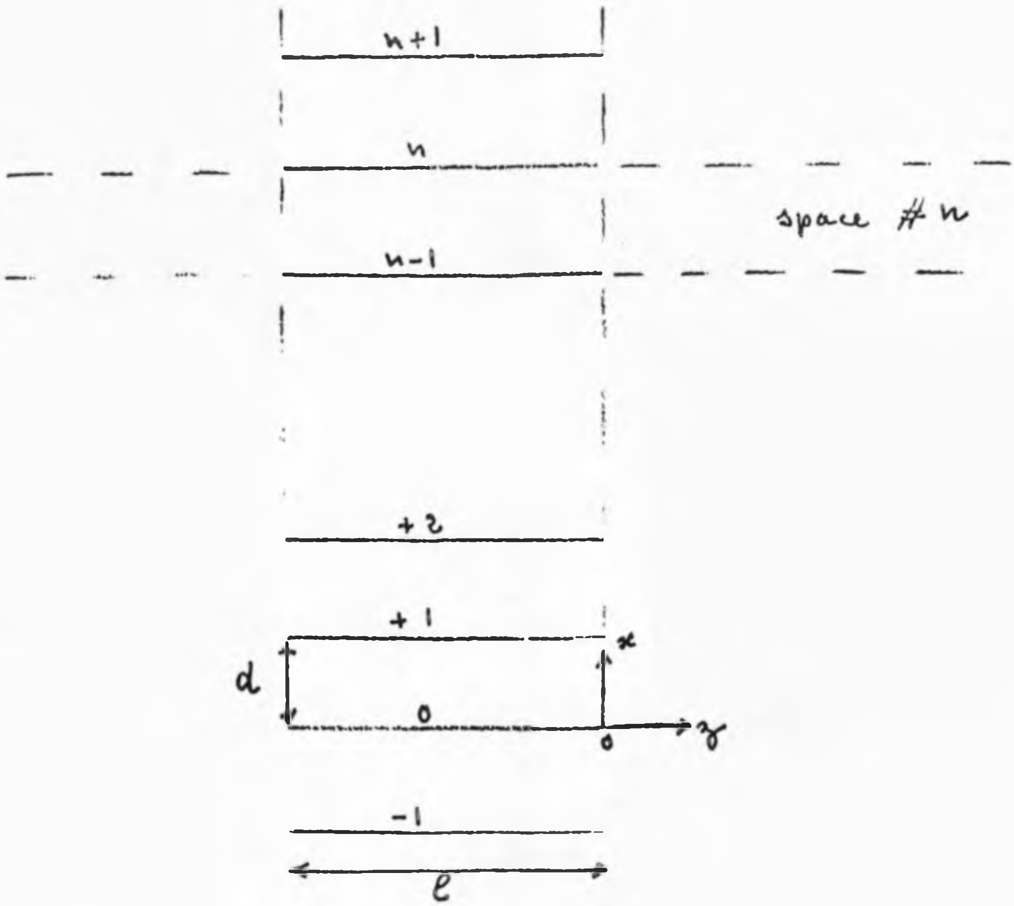
ABSTRACT

The propagation of electromagnetic waves along an infinite "corrugated surface" is investigated by means of integral equations and Fourier transform techniques.

Results are obtained which take into account the finite distance between the corrugations. In the E case, we obtain in quite a natural way results similar to those previously obtained by R. Hurd.



(Fig. 1.)

1. Introduction

The "corrugated line" is constituted by perfectly conducting strips of length ℓ; d is the spacing between two consecutive strips. The line is infinite in both directions: $x = \pm\infty$. (Fig. 1).

In this two-dimensional problem, we shall first treat the H case where the field components are (E_y, H_x, H_z) , and the field is harmonic. We shall write, to simplify the notations:

$$\begin{aligned} E_y &= u \\ H_z &= \frac{j}{\mu\omega}\frac{\partial u}{\partial x} \end{aligned} \qquad (1.1)$$

We shall omit everywhere the factor $e^{j\omega t}$.

At the edges $\begin{cases} x = nd \\ z = 0 \ , \ z = -\ell \end{cases}$ we need a condition to insure a unique solution. If r is the distance from the edge to a neighbor point, the edge condition is (Jones 1):

$$E = O(1) \qquad r \to 0$$

$$H_z = O(r^{-1}) \qquad r \to 0 \quad .$$

As shown on Fig. 1, the region $(n-1)d < x < nd$ is called region n . For that region, u_n and $\frac{\partial u_n}{\partial x}$ are the corresponding values of u and $\frac{\partial u}{\partial x}$. The Green functions we shall use satisfy:

$$\Delta_2 G_n + k^2 G_n = -\delta(x - x_o)\,\delta(z - z_o)$$

with $$k = \frac{2\pi}{\text{wavelength } \lambda} \quad .$$

G_n correspond to outgoing waves and boundary conditions are

$$G = 0 \quad \text{for} \quad \begin{cases} x = (n-1)d \\ x = nd \end{cases} \quad .$$

Applying Green's formula to region n we get

$$\begin{aligned} u_n(x,z) = &\int_{-\infty}^{+\infty} \frac{\partial G_n}{\partial x'}\left(x,(n-1)d,|z-z'|\right) u_n\left[(n-1)d,\ z'\right] dz' \\ &- \int_{-\infty}^{+\infty} \frac{\partial G_n}{\partial x'}\left(x,nd,|z-z'|\right) u_n(nd,z')\, dz' \end{aligned} \qquad (1.2)$$

or

$$\frac{\partial u_n}{\partial x}(x,z) = \int_{-\infty}^{+\infty} \frac{\partial^2 G_n}{\partial x\,\partial x'}\left(x,(n-1)d,|z-z'|\right) u_n\left[(n-1)d,\ z'\right] dz'$$

$$-\int_{-\infty}^{+\infty} \frac{\partial^2 G_n}{\partial x\,\partial x'}\,(x,nd,|z-z'|)\ u_n(nd,z')\ dz' \quad .$$

Putting then $x = nd - 0$, and $x = (n-1)d + 0$, we get:

$$\frac{\partial u_n}{\partial x}(nd-0,z) = \int_{-\infty}^{+\infty} \frac{\partial G_n}{\partial x'\partial x'}\left(nd,(n-1)d,|z-z'|\right) u_n\left[(n-1)d,z'\right]\ dz'$$

$$-\int_{-\infty}^{+\infty} \frac{\partial G_n}{\partial x'\partial x}(nd,nd,|z-z'|)\ u_n(nd,z')\ dz' \tag{1.3}$$

$$\frac{\partial u_n}{\partial x}\left[(n-1)d+0,z\right] = \int_{-\infty}^{+\infty} \frac{\partial^2 G_n}{\partial x'\,\partial x}\left[(n-1)d,(n-1)d,|z-z'|\right]\ u_n\left[(n-1)d,z'\right] dz'$$

$$-\int_{-\infty}^{+\infty} \frac{\partial G_n}{\partial x'\,\partial x}\left[(n-1)d,\ nd,\ |z-z'|\right] u_n(nd,z')dz' \quad . \tag{1.4}$$

We apply to that periodic structure Floquet's theorem. u is continuous everywhere so that:

$$\begin{aligned} e^{i\gamma d}\ u_n\left[(n-1)d+0,z\right] &= u_n\left[(nd-0,z)\right] \\ e^{i\gamma d}\ \frac{\partial u}{\partial x}n\left[(n-1)d+0,z\right] &= \frac{\partial u_n}{\partial x}\left[nd-0,z^2\right] \end{aligned} \tag{1.5}$$

and from (1.3) and (1.4) we get:

$$\int_{-\infty}^{+\infty} \left\{ \frac{\partial^2 G_n}{\partial x' \partial x} (nd,(n-1)d,|z-z'|)\, e^{-i\gamma d} - \frac{\partial G_n}{\partial x'\, \partial x} (nd,nd,|z-z'|) \right. \tag{1.6}$$

$$\left. -\frac{\partial G_n}{\partial x'\, \partial x} \left[(n-1)d,(n-1)d,|z-z'|\right] + \frac{\partial G_n}{\partial x'\, \partial x} \left[(n-1)d,nd,|z-z'|\right] e^{i\gamma d} \right\}$$

$$u(nd,z')\, dz' = 0 \quad \begin{cases} z < -\ell \\ z > 0 \end{cases} .$$

where we suppress the subscript n for u_n .

2. <u>Green function for the problem.</u>

We need only the Fourier transforms of $G_n(x,x',|z-z'|)$,

$$\int_{-\infty}^{+\infty} G_n(x,x',\zeta)\, e^{-s\zeta}\, d\zeta = g_n(x,x',s)$$

$$g_n(x,x',s) = \frac{\sin\left\{\sqrt{s^2+k^2}\,[x'-(n-1)d]\right\} \sin\left\{\sqrt{s^2+k^2}\,[x-nd]\right\}}{\sqrt{s^2+k^2}\ \sin \alpha\ \sqrt{s^2+k^2}} \qquad x \geq x'$$

$$\frac{\sin\left\{\sqrt{s^2+k^2}\ [x'-nd]\right\} \sin\left\{\sqrt{s^2+k^2}\ [x-(n-1)d]\right\}}{\sqrt{s^2+k^2}\, \sin d\ \sqrt{s^2+k^2}} \qquad x \leq x' . \tag{2.1}$$

If we assume that

$$k = kr - i\, k_i \qquad k_i > 0 ,$$

The above expression is valid for $\quad s = \sigma + i\tau \quad \ni$

$$-k_i < \sigma < +k_i .$$

We furthermore have:

$$\frac{\partial^2 G_n}{\partial x'\, \partial x} (nd,nd,s) = + \frac{\partial^2 G_n}{\partial x\, \partial x} \left[(n-1)d,(n-1)d,s\right] = - \frac{\cos \sqrt{s^2+k^2}\, d}{\sin \sqrt{s^2+k^2}\, d} \sqrt{s^2+k^2} \tag{2.2}$$

$$\frac{\partial^2 G_n}{\partial x' \, \partial x}\left[nd,(n-1)d,s\right] = \frac{\partial^2 G_n}{\partial x' \, \partial x}\left[(n-1)d,nd,s\right] = -\frac{\sqrt{s^2+k^2}}{\sin\sqrt{s^2+k^2}\,d} \quad . \qquad (2.3)$$

3. Fourier transform of the Integral Equation.

We then define:

$$v+(x,s) = \int_0^{\infty} u(x,z)\, e^{-sz}\, dz$$

$$v-(x,s) = \int_{-\infty}^{-\ell} u(x,z)\, e^{s(z+\ell)}\, dz$$ and we have: $v_+ = \pm v_-$ if

we consider the symmetric and antisymmetric case. Assuming those functions to be analytic for $Re(s) > -k_1$ we get if $-k_1 < c$,

$$u(x,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v+(x,s)\, e^{sz}\, ds \qquad z > 0$$

$$0 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v+(x,s)\, e^{sz}\, ds \qquad z < 0$$

$$u(x,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v-(x,s)\, e^{-s(z+\ell)}\, ds \qquad z < 0$$

$$0 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v-(x,s)\, e^{-s(z+\ell)}\, ds \qquad z > 0 \ .$$

So equation (1.6) may be written:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[v+(nd,s)e^{sz} + v-(nd,s)\, e^{-s(z+\ell)}\right] K(s)\, ds = 0 \ , \qquad (3.1)$$

with $K(s) = 2\,\dfrac{\left[\cos\sqrt{s^2+k^2}\,d - \cos\gamma d\right]}{\sin d\sqrt{s^2+k^2}}\,\sqrt{s^2+k^2}$.

4. Factorization of K(s)

We first factorize $\frac{\sqrt{s^2+k^2}}{\sin d\sqrt{s^2+k^2}} = H(s)$

$$\frac{\sqrt{s^2+k^2}}{\sin d\sqrt{s^2+k^2}} = \frac{1}{d\prod_1^\infty \left(1+\frac{sd}{\pi\lambda n}\right) e^{-\frac{sd}{\pi\lambda n}}} \left(\frac{kd}{\sin kd}\right) \tag{4.1}$$

with $\lambda_n = \sqrt{n^2 - (\frac{kd}{\pi})^2}$

We write then:

$$H_+(s) = \frac{1}{C_+\prod_1^\infty \left(1+\frac{sd}{\pi\lambda n}\right) e^{-\frac{sd}{\pi\lambda n}}} \tag{4.2}$$

$$H_-(s) = C_-\prod_1^\infty \left(1-\frac{sd}{\pi\lambda n}\right) e^{+\frac{sd}{\pi\lambda n}} \quad . \tag{4.3}$$

We have $\frac{1}{C_+ C_-} = \frac{k}{\sin kd}$

and to get $\quad H_+(s)\, H_-(-s) = -1 \quad ,$

we need $\quad C_-/C_+ = -1$

so that $\quad C_+ = \sqrt{-\frac{\sin kd}{k}}$

$$C_- = -\sqrt{-\frac{\sin kd}{k}} \quad .$$

In the same way we have to factorize:

$$\cos kd - \cos \gamma d = \frac{M_+}{M_-} \quad . \tag{4.4}$$

The zeros are given by

$$s = \pm\sqrt{(\gamma + \frac{2n\pi}{d})^2 - k^2} = \mu_n \qquad n = -\infty \cdot\cdot\cdot +\infty$$

so we can write:

$$\cos kd - \cos\gamma d = (\cos kd - \cos\gamma d)\prod_{-\infty}^{+\infty}{}' \left(1 + \frac{sd/2\pi}{\sqrt{\left[(\gamma\frac{d}{2\pi}) + n\right]^2 - (\frac{kd}{2\pi})^2}}\right)$$

$$\prod_{-\infty}^{+\infty}{}' \left(1 - \frac{sd/2\pi}{\sqrt{\left[(\gamma\frac{d}{2\pi}) + n\right]^2 - (\frac{kd}{2\pi})^2}}\right)$$

$$M_+ = D_+\left[1 + \frac{sd/2\pi}{\sqrt{(\frac{\gamma d}{2\pi})^2 - (\frac{kd}{2\pi})^2}}\right]\prod_{1}^{\infty}\left[\left(1 + \frac{sd/2\pi}{\sqrt{(n + \frac{\gamma d}{2\pi})^2 - (\frac{kd}{2\pi})^2}}\right)e^{-\frac{sd/2\pi}{(n + \frac{\gamma d}{2\pi})}}\right]$$

$$\prod_{1}^{\infty}\left(1 + \frac{sd/2\pi}{\sqrt{(n - \frac{\gamma d}{2\pi})^2 - (\frac{kd}{2\pi})^2}}\right) e^{-\frac{sd/2\pi}{(n - \frac{\gamma d}{2\pi})}} \tag{4.5}$$

$$M_- = \frac{1}{D_-\left[1 - \frac{sd/2\pi}{\sqrt{(\frac{\gamma d}{2\pi})^2 - (\frac{kd}{2\pi})^2}}\right]\prod_{1}^{\infty}\left(1 - \frac{-sd/2\pi}{\sqrt{(n + \frac{\gamma d}{2\pi})^2 - (\frac{kd}{2\pi})^2}}\right) e^{+\frac{sd/2\pi}{(n + \frac{\gamma d}{2\pi})}}}$$

$$\prod_{1}^{\infty}\left(1 - \frac{sd/2\pi}{(n - \frac{d}{2\pi})^2 - (\frac{kd}{2\pi})^2}\right) e^{\frac{sd/2\pi}{(n - \frac{\gamma d}{2\pi})}} \tag{4.6}$$

and we have: $\quad D_+ + D_- = (\cos kd - \cos\gamma d)$.

To get $\frac{D_+}{D_-} = -1$, we have to take

$$D_+ = \sqrt{\cos \gamma d - \cos kd}$$
$$D_- = -\sqrt{\cos \gamma d - \cos kd}$$

5. Asymptotic Values of the Infinite Products

We shall compare H_+ to $\Gamma(\frac{sd}{\pi})$ for $|\text{Argt } s| \leq \frac{\pi}{2}$.

$$\frac{H_+(s)}{\Gamma(\frac{sd}{\pi})} = \frac{1}{C_+} \frac{\prod_1^\infty (1+\frac{sd}{\pi n}) e^{-\frac{sd}{\pi n}} \cdot \frac{sd}{\pi} e^{\gamma_0 \frac{sd}{\pi}}}{\prod_1^\infty (1+\frac{sd}{\pi \lambda n}) e^{-\frac{sd}{\pi n}}} \qquad \gamma_0 = \text{Euler constant}$$

(5.1)

$$= \frac{sd}{\pi C_+} e^{\gamma_0 \frac{sd}{\pi}} \prod_1^\infty \frac{(1+\frac{sd}{\pi n})}{(1+\frac{sd}{\pi \lambda n})} .$$

Putting $\frac{sd}{\pi} = \frac{1}{\zeta}$, we have:

$$\prod_1^\infty \frac{(1+\frac{sd}{\pi n})}{(1+\frac{sd}{\pi \lambda n})} = \frac{\prod_1^\infty (\frac{\lambda n}{n})}{\prod_1^\infty \left[1+\frac{(\lambda n - n)}{1+\zeta n}\right]} = N(\zeta) \sqrt{\prod_1^\infty (\frac{\lambda n}{n})}$$

$f_n(\zeta) = -\frac{\zeta(\lambda n - n)}{1+\zeta n}$, $N(\zeta)$ converges uniformly in $|\text{Argt } \zeta| \leq \frac{\pi}{2}$

if the series $\sum_1^\infty |f_n(\zeta)|$ does the same.

For large n

$$|f_n(\zeta)| \leq \frac{|\zeta| \cdot |(\frac{kd}{\pi})^2 - \varepsilon_n|}{n\,|1+n\zeta|} \qquad \varepsilon_n \to 0 \quad \text{when} \quad n \to \infty$$

and for $|\text{Argt } \zeta| \leq \frac{\pi}{2}$ $\qquad |1+n\zeta| \geq n|\zeta|$.

Hence, $$H_n(\zeta) \leq \frac{\left|(\frac{kd}{\pi})^2 - \varepsilon_n\right|}{n^2} \quad .$$

Thus the series converges uniformly and $N(\zeta) \to 1$ when $\zeta \to 0$.

Moreover, it can be shown that: $N(\zeta) = 1 + O(\zeta \log \zeta)$.

We have,

$$\prod_1^{\infty} \frac{\lambda n}{n} = \sqrt{\prod_1^{\infty} \left[\frac{n^2 - (\frac{kd}{\pi})^2}{n^2}\right]} = \sqrt{\prod_1^{\infty} 1 - (\frac{kd}{\pi n})^2} = \sqrt{\frac{\sin kd}{kd}} \quad .$$

So $$\frac{H+(s)}{\Gamma(\frac{sd}{\pi})} = \frac{is\sqrt{d}}{\pi}\, e^{\gamma_0 \frac{sd}{\pi}} \left(1 + O(\frac{\log s}{s})\right) \quad .$$

In the same way we shall compare M_+ to $\Gamma(\frac{sd}{2\pi} + \frac{d}{2\pi} + 1)\ \Gamma(\frac{sd}{2\pi} - \frac{\gamma d}{2\pi} + 1)$.

$$M_+ \Gamma(\frac{sd}{2\pi} + \frac{\gamma d}{2\pi} + 1)\Gamma(\frac{sd}{2\pi} - \frac{\gamma d}{2\pi} + 1) = \sqrt{\cos \gamma d - \cos kd}\left[1 + \frac{sd/2\pi}{(\frac{\gamma d}{2\pi})^2 - (\frac{kd}{2\pi})^2}\right] \times$$

$$\left\{\frac{\prod_1^{\infty}\left(1 + \frac{sd/2\pi}{(n + \frac{\gamma d}{2\pi})^2 - (\frac{kd}{2\pi})^2}\right) e^{-\frac{sd/2\pi}{(n + \frac{\gamma d}{2\pi})}}}{\prod_1^{\infty}\left(1 + \frac{sd/2\pi}{n + \frac{\gamma d}{2\pi}}\right) e^{-\frac{sd/2\pi}{(n + \frac{\gamma d}{2\pi})}}}\right\} \times \left\{\frac{\prod_1^{\infty}\left(1 + \frac{sd/2\pi}{(n - \frac{d}{2\pi})^2 - (\frac{kd}{2\pi})^2}\right) e^{-\frac{sd/2\pi}{(n - \frac{\gamma d}{2\pi})}}}{\prod_1^{\infty}\left(1 + \frac{sd/2\pi}{n - \frac{\gamma d}{2\pi}}\right) e^{-\frac{sd/2\pi}{(n - \frac{\gamma d}{2\pi})}}}\right\} \quad (5.2$$

$$\times\ \Gamma(\frac{\gamma d}{2\pi} + 1)\Gamma(-\frac{\gamma d}{2\pi} + 1)\, e^{\frac{sd}{2\pi}\left[\psi(\frac{\gamma d}{2\pi} + 1) + \psi(-\frac{\gamma d}{2\pi} + 1)\right]} \quad .$$

We can use the same convergence discussion as before. The first $\{\ \}$ can be written as, putting $\frac{sd}{2\pi} = \frac{1}{\zeta}$,

$$\mu_n \gamma^+ = \sqrt{(n+\frac{\gamma d}{2\pi})^2 - (\frac{kd}{2\pi})^2}$$

$$\mu_n \gamma^- = \sqrt{(n-\frac{\gamma d}{2\pi})^2 - (\frac{kd}{2\pi})^2}$$

$$\frac{\prod_1^\infty \left[1 + \frac{\zeta\left[\mu_n, \gamma_+ - (n+\frac{\gamma d}{2\pi})\right]}{1 + (n+\frac{\gamma d}{2\pi})\zeta}\right]}{\prod_1^\infty \left(\frac{\mu_n, \gamma_+}{n + \frac{\gamma d}{2\pi}}\right)} .$$

One can easily see that for $|\mathrm{Argt}\,\zeta| \leq \frac{\pi}{2}$ the numerator $= 1 + O(\zeta \log \zeta)$,

and the denominator:

$$\sqrt{\prod_1^\infty \left[1 - \frac{(\frac{kd}{2\pi})^2}{(n+\frac{\gamma d}{2\pi})^2}\right]} .$$

The other bracket gives us the product

$$\sqrt{\prod_1^\infty \left[1 - \frac{(\frac{kd}{2\pi})^2}{(n-\frac{\gamma d}{2\pi})^2}\right]}$$

and we have

$$\left[1 - \frac{(\frac{kd}{2\pi})^2}{(\frac{\gamma d}{2\pi})^2}\right] \prod_{-\infty}^{+\infty}{}' \left[1 - \frac{(\frac{kd}{2\pi})^2}{(n+\frac{\gamma d}{2\pi})^2}\right] = \frac{\cos kd - \cos \gamma d}{1 - \cos \gamma d} .$$

So for large values of s

$$M_+ \Gamma(\frac{sd}{2\pi} + \frac{\gamma d}{\pi} + 1) \Gamma(\frac{sd}{2\pi} - \frac{\gamma d}{2\pi} + 1) \cong$$

$$\Gamma(\frac{\gamma d}{2\pi}+1)\Gamma(-\frac{\gamma d}{2\pi}+1)\; e^{\frac{sd}{2\pi}\left(\psi(\frac{\gamma d}{2\pi}+1) + \psi(-\frac{\gamma d}{2\pi}+1)\right)} \frac{is}{\gamma}\sqrt{1-\cos \gamma d} \tag{5.3}$$

$$M_+ H_+ = \frac{is}{\gamma} \cdot \Gamma(1+\frac{\gamma d}{2\pi})\Gamma(1-\frac{\gamma d}{2\pi})\, e^{\frac{sd}{2\pi}\left[\psi(1+\frac{\gamma d}{2\pi}) + \psi(1-\frac{\gamma d}{2\pi})\right]}$$

$$\frac{is\sqrt{d}}{\pi} \cdot e^{\gamma\frac{sd}{\pi}} \left[\frac{\Gamma(\frac{sd}{\pi})}{\Gamma(\frac{sd}{2\pi}+\frac{\gamma d}{2\pi}+1)\Gamma(\frac{sd}{2\pi}-\frac{\gamma d}{2\pi}+1)}\right] \sqrt{1-\cos\gamma d}$$

and

$$\frac{\Gamma(2\,\frac{sd}{2\pi})}{\Gamma(\frac{sd}{2\pi}+\frac{\gamma d}{2\pi}+1)\Gamma(\frac{sd}{2\pi}-\frac{\gamma d}{2\pi}+1)} = \frac{\frac{1}{\sqrt{2\pi}}\cdot 2^{(\frac{sd}{\pi}-\frac{1}{2})}\,\Gamma(\frac{sd}{2\pi})\Gamma(\frac{sd}{2\pi}+\frac{1}{2})}{\Gamma(\frac{sd}{2\pi}+\frac{\gamma d}{2\pi}+1)\quad\Gamma(\frac{sd}{2\pi}-\frac{\gamma d}{2\pi}+1)}$$

$$= \frac{1}{2\sqrt{\pi}} \cdot e^{\frac{sd}{\pi}\log 2} \quad (\frac{sd}{2\pi})^{-(1+\frac{\gamma d}{2\pi})-\frac{1}{2}+\frac{\gamma d}{2\pi}}$$

so

$$M_+ H_+ \cong C \cdot \sqrt{s} \cdot e^{\frac{sd}{2\pi}\left[\psi(1+\frac{\gamma d}{2\pi}) + \psi(1-\frac{\gamma d}{2\pi}) + 2\gamma_0 + 2\log 2\right]}$$

$$\cong C\sqrt{S}\, e^{\frac{sd}{2\pi}\left[\psi(1+\frac{\gamma d}{2\pi}) + \psi(1-\frac{\gamma d}{2\pi}) + 2\gamma_0 + 2\log 2\right]}$$

with $C = -\dfrac{\sqrt{d}}{\pi\gamma\cdot 2\sqrt{\pi}} \cdot (\dfrac{d}{2\pi})^{-3/2} \sqrt{1-\cos\gamma d} \cdot \dfrac{\gamma d}{2\pi} \cdot \dfrac{\pi}{\sin\frac{\gamma d}{2}}$

$$= -\frac{\sqrt{2}}{\gamma d}\sqrt{\frac{1-\cos\gamma d}{2\sin^2\frac{\gamma d}{2}}} \cdot \frac{\gamma d}{\sqrt{2}} = -1$$

and we shall define

$$(M_+ H_+)^* = M_+ H_+\, e^{-\frac{sd}{2\pi}\left[\psi(1+\frac{\gamma d}{2\pi}) + \psi(1-\frac{\gamma d}{2\pi}) + 2\gamma_0 + 2\log 2\right]} = -1\sqrt{S} \,. \tag{5.4}$$

In the following we shall write:

$$(M_+ H_+)^* = K_+^* \; . \tag{5.5}$$

In the same way one can see easily that

$$K_-^* = \frac{-1}{\sqrt{S}} \; .$$

6. Solution of the Integral Equation.

If we are looking for a symmetric solution:

$v = v_+ = v_-$ and we have:

$$\int_{c-i\infty}^{c+i\infty} v \frac{K_+^*}{K_-^*} \left[e^{sz} + e^{-s(z+\ell)} \right] ds = 0 \; . \qquad z > 0 \, , \; z < -\ell . \tag{6.1}$$

The integral $\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sz}}{K_+^*(S)} \, ds$ exists, and is uniformly convergent for all values of z except 0 and vanishes for $z < 0$. Let the value for $z > 0$ be $\mathcal{K}_+(z)$. From well-known properties of the Laplace transform, we have:

$$\frac{1}{K_+^*(S)} = \int_0^\infty \mathcal{K}_+(z) \, e^{-sz} \, dz \; . \qquad \sigma > -k_1 \; . \tag{6.2}$$

If we rewrite our integral equation as:

$$\int_{c-i\infty}^{c+i\infty} v \frac{K_+^*}{K_-^*} \left[e^{s(z+\zeta)} + e^{-s(z+\zeta+\ell)} \right] ds = 0 \; . \tag{6.3}$$

Multiplying by $\mathcal{K}_+(\zeta)$ and integrating between 0 and ∞, noticing that $\int_{c-i\infty}^{c+i\infty} v \, K_+^* \, e^{sz} \, ds = 0$, $z < 0$, and $K_+^*(S) \, K_-^*(-S) = -1$, we get

$$\boxed{v(s)\,K_+^{*}(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{v\,e^{-w\ell}\,dw}{(w+s)K_-^{*}(w)}} \qquad \sigma + R(w) > 0\,. \tag{6.4}$$

Remembering that $v(s) = 0\,(\frac{1}{s^{1+\alpha}})$, we may apply Cauchy's residue theorem using the contour $\mathscr{C}$ below, Figure 2.

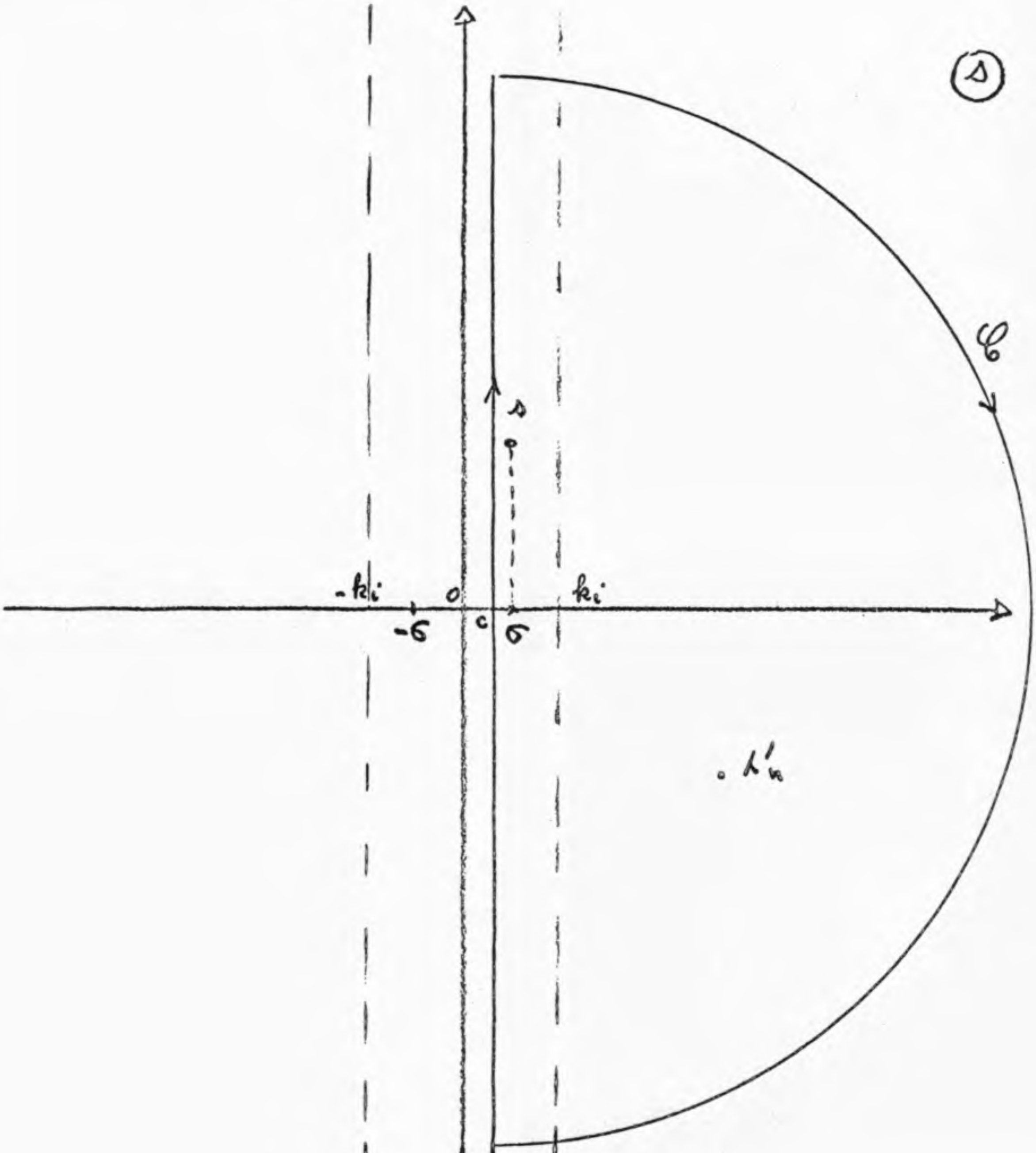


Figure 2.

Contribution from the large circle is zero.

$$K^{\#}_{-}(w) = 0(\frac{1}{\sqrt{w}})$$

$$v(w) = 0(\frac{1}{w^{1+\alpha}}) \quad \alpha > 0 \ .$$

The only poles come from $w = \lambda'n$, $\lambda'n = \frac{\pi}{d}\ \lambda n$, so we get:

$$v(s)\ K^{\#}_{+}(s) = \sum_{n=1}^{\infty} \frac{v(\lambda' n)\ e^{-\lambda'_n \ell}}{(s+\lambda' n)\ \overline{K}^{\#}_{-}(\lambda' n)} \tag{6.5}$$

where $\overline{K}^{\#}_{-}(\lambda' n) = M^{\#}_{-}(\lambda' n)\ H'^{\#}_{-}(\lambda' n)$. (6.6)

So to determine γ we get the infinite linear system

$$\sum_{n} v(\lambda_n)\left[\ K^{\#}_{+}(\lambda' n)\ \delta_{n\gamma} - \frac{e^{-\lambda' n \ell}}{(\lambda'_\gamma + \lambda' n)\ \overline{K}^{\#}_{-}(\lambda' n)}\right] = 0 \ ; (\delta_{n\gamma} : \text{Kronecker } \delta) \tag{6.7}$$

or if we take

$t = \frac{v}{\overline{K}^{\#}_{-}}$ as an unknown, we must have:

$$\left|\left[K^{\#}_{+}(\lambda' n)\ \overline{K}^{\#}_{-}(\lambda' n) \cdot \delta_{\gamma n} - \frac{e^{-\lambda' n \ell}}{\lambda'_\gamma + \lambda' n}\right]\right| = 0 \ . \tag{6.8}$$

7. Discussion of the Infinite Linear System.

Let us suppose now k real and

$$n_o > \frac{kd}{\pi} > n_o - 1 \ .$$

So for $n \geq n_o$, λ'_n is real and the nondiagonal terms in the infinite determinant are $0(\frac{e^{-\ell n}}{n})$.

For $n < n_o$, the corresponding terms would be $O(\frac{1}{n})$. The principal term is the main diagonal, so equating this term to zero

$$\boxed{K_+^{\#}(\lambda' n)\, K_-^{\#}(\lambda' n) = \frac{e^{-\ell\lambda' n}}{2\lambda' n} \qquad n = 1, 2, \ldots .} \tag{7.1}$$

Let us suppose for instance, $n_o = 1$; λ'_1 only is purely imaginary; all others λ_n are real; let us suppose γ real $> k$, then

$$|K_+^{\#}(s)| = |K_+(s)|$$

$$|K_-^{\#}(s)| = |K_-(s)|$$

and

$$|K_+(s)|\,|\bar{K}_-(s)| = \prod_1^{\infty}\left[\frac{\left|1-\frac{sd}{\pi\lambda_n}\right|\left|1+\frac{sd}{2\pi\mu n\gamma+}\right|\left|1+\frac{sd}{2\pi\mu n\gamma-}\right|}{\left|1+\frac{sd}{\pi\lambda_n}\right|\left|1-\frac{sd}{2\pi\mu n\gamma-}\right|\left|1-\frac{sd}{2\pi\mu n\gamma-}\right|}\right]$$

$$\times\frac{\left|\frac{d}{\pi\lambda_1}\right|}{\left|1+\frac{sd}{\pi\lambda_1}\right|}\prod_1^{\infty}\left[\frac{\left|1+\frac{sd}{2\pi\mu\gamma+}\right|\left|1+\frac{sd}{2\pi\mu\gamma_-}\right|}{\left|1-\frac{sd}{2\pi\mu\gamma+}\right|\left|1-\frac{sd}{2\pi\mu\gamma_-}\right|}\right]\times\frac{\left|1+\frac{sd}{2\pi\mu_o\gamma_+}\right|\left|1+\frac{sd}{2\pi\mu_o\gamma_-}\right|}{\left|1-\frac{sd}{2\pi\mu_o\gamma_+}\right|\left|1-\frac{sd}{2\pi\mu_o\gamma_-}\right|} \tag{7.2}$$

so $$|K_+(\lambda'_1)|\,|\bar{K}_-(\lambda'_1)| = \frac{1}{2\lambda'_1} \tag{7.3}$$

and this gives a real value of ℓ.

For practical purposes, we shall use equation (7.1) as follows: Putting $\gamma > k$ in the first member, we deduce from that the value of ℓ equating the arguments of both sides. So we get:

$$\ell\cdot\lambda' = \phi + 2h\pi \qquad h \text{ integer}$$

$$\boxed{\ell = \frac{\phi}{\lambda'_1} + \frac{2h\pi}{\lambda'_1}}$$

so, for $\lambda'_1 \neq k$,

$$\ell \neq \frac{\phi}{\lambda'_1} + \lambda .$$ λ free space wavelength

So we have "passing bands" for values of ℓ, differing by a constant length which is, in general $\neq \lambda$.

Remark:

$$\lim_{\to s} \lambda_n \frac{K_+^*(s)}{K_-^*(s)} = \frac{K_+^*(s)}{\left[K_-^*(s)\right]'_{s\to\lambda_n}} = \left[\frac{[\cos Kd - \cos \gamma d]\ K}{\frac{s}{K}\, d \cos Kd}\right]_{s=\lambda_n}$$

$$= \frac{1-(-1)^n \cos \gamma d}{\lambda_n} \frac{\pi n^2}{d^2}$$

so our formula becomes:

$$\frac{\left[K_+^*(\lambda'_n)\right]^2 \frac{d^2\lambda_n}{\pi n^2}}{1-(-1)^n \cos \gamma d} = \frac{e^{-\ell\lambda'_n}}{2\lambda'_n} \qquad n = 1 \cdots \tag{7.5}$$

which may be more practical for computation.

The E Case

1. Introduction

We now consider the E case where the field components are (E_x, E_z, H_y) with harmonic waves.

To simplify the notations, we shall write

$$\begin{aligned} H_y &= u \\ E_z &= \frac{1}{j\omega\varepsilon} \frac{\partial u}{\partial x} \end{aligned} \qquad . \tag{1.1}$$

We shall omit everywhere the factor $e^{j\omega t}$.

At the edges $\begin{cases} x = nd \\ z = 0 \end{cases}$ $\begin{cases} x = nd \\ z = -\ell \end{cases}$ we need a condition to insure a unique solution. If r is the distance from the edge to a neighbor point, the edge condition is: $u = O(1)$ and from Maxwell's equations, it would result $\frac{\partial u}{\partial x} = O(r^{-1})$.

As shown on Fig. 1, the region $(n-1)d \leq x \leq nd$ is called region n . For region n we call u_n and $\frac{\partial u_n}{\partial x}$ the corresponding values of u and $\frac{\partial u}{\partial x}$. We shall use Green functions satisfying

$$\nabla^2 G_n + k^2 G_n = -\delta(x - x_0) \quad \delta(z - z_0)$$

$$k = 2\pi/\text{wavelength}$$

G corresponds to outgoing waves, and boundary conditions are:

$$\frac{\partial G_n}{\partial x} = 0 \qquad \text{for } x = nd \text{ and } x = (n-1)d \ .$$

Applying Green's formula to region n we get:

$$u_n(x,z) = -\int_{-\infty}^{+\infty} G_n(x,(n-1)d,|z-z'|) \left.\frac{\partial u_n}{\partial x'}\right|_{(n-1)d+0} + \int_{-\infty}^{+\infty} G_n(x,nd,|z-z'|) \left.\frac{\partial u_n}{\partial x'}\right|_{nd-0} dz' \tag{1.2}$$

The two preceding equations give:

$$u_n(nd-0,z) = -\int_{-\infty}^{+\infty} G_n(nd,(n-1)d,|z-z'|)\,\frac{\partial u_n}{\partial x'}(n-1)d,z')dz' \qquad (1.3)$$

$$+\int_{-\infty}^{+\infty} G_n(nd,nd,(z-z')\;\frac{\partial u_n}{\partial x'}(nd,z')\;dz'$$

$$u_n\left[(n-1)d,z\right] = -\int_{-\infty}^{+\infty} G_n\left[(n-1)d,(n-1)d,|z-z'|\right]\frac{\partial u_n}{\partial x'}\;(n-1)d,z')dz'$$

$$+\int_{-\infty}^{+\infty} G_n\left[(n-1)d,nd,|z-z'|\right]\frac{\partial u_n}{\partial x'}\;(nd,z')dz' \;. \qquad (1.4)$$

Applying now Floquet's theorem to this periodic structure we get:

$$e^{i\gamma d}\,u_n\left[(n-1)d,z\right] = u_{n+1}\left[nd,z\right] \;.$$

$$e^{i\gamma d}\,\frac{\partial u_n}{\partial x'}\left[(n-1)d,z\right] = \frac{\partial u_n}{\partial x'}\left[nd,z\right] \qquad (1.5)$$

from continuity of $\frac{\partial u}{\partial x'}$.

So we get finally the integral equation:

$$u_{n+1}(nd,z)-u_n(nd,z) = +\int_{-\infty}^{+\infty}\Big\{G_n\left[nd,(n-1)d,|z-z'|\right]e^{-i\gamma d}$$

$$-G_n(nd,nd,|z-z'|)\Big\}\,\frac{\partial u}{\partial x'}\,(nd,z')dz' \;-\int_{-\infty}^{+\infty}\Big\{G_n\left[(n-1)d,(n-1)d_1|z-z'|\right]$$

$$-G_n\left[(n-1)d,nd,|z-z'|\right]\Big\}\frac{\partial u}{dx'}\,(nd,z')dz' \,. \qquad (1.6)$$

We shall call the first member $\delta u_n(nd,z)$.

2. Transformed Integral Equation

We shall now define:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v_+(s)\, e^{sz}\, ds = \begin{cases} \dfrac{\partial u}{\partial x}(nd,z) & z>0 \\ 0 & z<0 \end{cases}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v_-(s)\, e^{-s(z+\ell)}\, ds = \begin{cases} 0 & z>-\ell \\ \dfrac{\partial u}{\partial x}(nd,z) & z<-\ell \end{cases}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g_n(x,x',s)\, e^{sz}\, ds = G_n(x,x',|z-z'|) \qquad -\infty < z < +\infty$$

c is real and $|c| < k_1$.

And we have

$$g_n = \int_{-\infty}^{+\infty} G_n(x,x',\zeta)\, e^{-s\zeta}\, d\zeta \quad .$$

It can easily be shown that:

$$g_n(x,x',s) = -\frac{\cos\left\{\sqrt{s^2+k^2}\cdot\left[x'-(n-1)d\right]\right\}\cos\left\{\sqrt{s^2+k^2}\left[x-nd\right]\right\}}{\sqrt{s^2+k^2}\ \sin d\sqrt{s^2+k^2}} \qquad x \geq x'$$

$$\qquad\qquad\qquad (2.1)$$

$$-\frac{\cos\sqrt{s^2+k^2}\,(x'-nd)\,\cos\sqrt{s^2+k^2}\left[x-(n-1)d\right]}{\sqrt{s^2+k^2}\ \sin d\sqrt{s^2+k^2}} \qquad x \leq x' .$$

We have then:

$$g_n(nd,nd,s) = -\frac{\cos\sqrt{s^2+k^2}\ d}{\sqrt{s^2+k^2}\ \sin\sqrt{k^2+s^2}\ d}$$

$$\qquad\qquad\qquad (2.2)$$

$$g_n\left[(n-1)d,(n-1)d,s\right] = -\frac{\cos\sqrt{s^2+k^2}\ d}{\sqrt{s^2+k^2}\ \sin\sqrt{k^2+s^2}\ d} \quad .$$

$$g_n\left[(n-1)d, nd, s\right] = -\frac{1}{\sqrt{s^2+k^2}\,\sin\sqrt{k^2+s^2}\,d} = g_n\left[nd,(n-1)d,\ s\right]. \tag{2.3}$$

As expected, these expressions are independent of n. We now write equation (1.6), taking into account the continuity of u for $x = nd$, $z < -\ell$ or $z > 0$

$$0 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[v_+\, e^{sz} + v_-\, e^{-s(z+\ell)}\right]\frac{2(\cos Kd - \cos\gamma d)\, ds}{K\sin Kd} \tag{2.4}$$

with $K = \sqrt{k^2+s^2}$.

We call

$$K(s) = \frac{\cos Kd - \cos\gamma d}{K\sin Kd}\,.$$

3. Factorization of the Kernel

As in the H case, we define $L_+^*(s)$ and $L_-^*(s)$. It is obvious here that:

$$L_+^*(s) = \frac{K_+^*(s)}{s+ik} \tag{3.1}$$

$$L_-^*(s) = (s-ik)\,K_-^*(s)\ , \tag{3.2}$$

and

$$L_+^*(s) = -\frac{1}{(s+ik)\sqrt{s}}\left[1+O\left(\frac{\log s}{s}\right)\right]$$

$$L_-^*(s) = -(s-ik)\,\sqrt{s}\left[1+O\,\frac{\log s}{s}\right].$$

4. Solution of the Integral Equation

By the same process as before we get

$$v(\lambda n)\left[L_+^*(\lambda' n)\,\delta_{n\nu} - \frac{e^{-\ell\lambda'_n}}{(\lambda_\nu' + \lambda'_n)\,L_-^*(\lambda' n)}\right] = 0 \tag{4.1}$$

$$n, \nu = 0, 1, \cdots .$$

Taking $z = \frac{v}{L_-^*}$ the infinite system determining γ becomes:

$$\left| L_+^*(\lambda' n)\ \bar{L}_-(\lambda' n)\ \delta_{n\nu} - \frac{e^{-\ell\lambda'_n}}{\lambda'_\nu + \lambda'_n} \right| = 0\ . \tag{4.2}$$

Suppose now $\frac{kd}{\pi} < 1$. The term corresponding to $\lambda_0 = ik$ is now:

$$\frac{1}{2ik} K_+^*(\lambda_0)\ K_-^*(\lambda_0) - \frac{e^{-\ell\lambda_0}}{2ik} = 0$$

or

$$e^{-ik_0} = K_+^*(ik_0)\ K_-^*(ik_0)\ . \tag{4.3}$$

As before, we can show that the modulus of the second member is one for values of $\gamma > k_0$; we then get real values for ℓ , and ℓ is defined by

$$\ell = \ell_1 + p\lambda\ ,$$

ℓ_1 being one of the solutions, p integer.

But:
$$\frac{K_+^*(s)}{K_-^*(s)} = \frac{(\cos Kd - \cos \gamma d)\ K}{\sin Kd}$$

$$\lim_{s \to ik_0} \frac{K_+^*(s)}{K_-^*(s)} = \frac{1}{d}(1 - \cos\gamma d)$$

$$e^{-ik_0\ell} = \frac{d}{1 - \cos\gamma d}\left[K_+^*(ik_0)\right]^2$$

Hence
$$\boxed{e^{-\frac{ik_0\ell}{2}} = \sqrt{\frac{d}{1 - \cos\gamma d}}\ K_+^*(ik_0)} \tag{4.4}$$

with

$$K_+^{\#}(s) = A\, e^{B\frac{sd}{2\pi}}$$

$$\times \frac{\prod_1^\infty \left(1+\frac{sd}{2\pi\mu_{n\gamma+}}\right) e^{-\frac{sd}{2\pi(n+\frac{\gamma d}{2\pi})}} \prod_1^\infty \left(1+\frac{sd}{2\pi\mu_{n\gamma-}}\right) e^{-\frac{sd}{2\pi(n-\frac{\gamma d}{2\pi})}} \left(1+\frac{sd}{\sqrt{\gamma^2-k^2}}\right)}{\prod_1^\infty \left(1+\frac{sd}{\pi\lambda_n}\right) e^{-\frac{sd}{\pi\lambda_n}}} \tag{4.5}$$

with

$$A = \sqrt{\frac{k(\cos kd - \cos \gamma d)}{\sin kd}}$$

$$B = -\left[\psi(1+\frac{\gamma d}{2\pi}) + \psi(1-\frac{\gamma d}{2\pi}) + 2\gamma + 2\log 2\right].$$

5. Approximate solution for the field.

If in the infinite linear system we neglect the terms containing $e^{-\ell\lambda_\nu}$ for $\nu \neq 0$, it remains:

$$\frac{v(\lambda'_0)e^{-\ell\lambda'_0}}{\lambda'_0 + \lambda'_\nu} + v(\lambda'_\nu)\, L_+^{\#}(\lambda'_\nu) = 0 . \qquad \nu = 1,2 \cdots \tag{5.1}$$

which gives an approximate solution for the $v(\lambda'_\nu)$

$$v(\lambda'_\nu) = \frac{v(\lambda'_0)\, e^{-\ell\lambda'_0}}{(\lambda'_0+\lambda'_\nu)L_+^{\#}(\lambda'_\nu)} . \tag{5.2}$$

If we consider now

$$v(s) = \frac{v(\lambda'_0)\, e^{-\ell\lambda'_0}}{(\lambda'_0+s)L_+^{\#}(s)} \tag{5.3}$$

This function is analytic in the right half plane and grows like $1/\sqrt{s}$. Hence it may be considered as an approximate solution for the field.

6. Computation of the propagation constant.

$$e^{ik\frac{\ell}{2}} = \sqrt{\frac{d}{1-\cos\gamma d}}\; K_+^{*}(ik_o) \quad . \tag{6.1}$$

Taking the argument of both sides we get, putting $\frac{kd}{2\pi} = K$, $\frac{\gamma d}{2\pi} = \Gamma$

$$\frac{k\ell}{2} = \sum\nolimits_1 + \sum\nolimits_2 + T - \sum\nolimits_3 , \tag{6.2}$$

with:

$$\sum\nolimits_1 = \sum_{n=1}^{\infty} \left[\text{Arctg} \frac{K}{\sqrt{(n+\Gamma)^2 - K^2}} - \frac{K}{n+\Gamma} \right]$$

$$\sum\nolimits_2 = \sum_{n=1}^{\infty} \left[\text{Arctg} \frac{K}{\sqrt{(n-\Gamma)^2 - K^2}} - \frac{K}{n-\Gamma} \right]$$

$$\sum\nolimits_3 = \sum_{n=1}^{\infty} \left[\text{Arctg} \frac{2K}{\sqrt{n^2 - 4K^2}} - \frac{2K}{n} \right]$$

$$T = \text{Arctg} \frac{K}{\sqrt{\Gamma^2 - K^2}} - \left[2\gamma_o + 2\log 2 + \psi(1+\Gamma) + \psi(1-\Gamma) \right] K$$

γ_o = Euler constant

ψ = logarithmic derivative of the gamma function.

We have supposed before $\Gamma > K$. Notice that

$$\sum\nolimits_1 + \sum\nolimits_2 + \text{Arctg} \frac{K}{\sqrt{\Gamma^2 - K^2}} = \sum_{-\infty}^{+\infty} \text{Arctg} \frac{K}{\sqrt{(\Gamma+n)^2 - K^2}}$$

$$+ \sum_{1}^{\infty} \left[\frac{K}{\Gamma - n} - \frac{K}{\Gamma + n} \right]$$

and

$$K\left[\psi(1+\Gamma+N)+\psi(1-\Gamma-N)\right] = \sum_{\nu=1}^{N} \frac{1}{\Gamma+\nu+1} - \sum_{\nu=0}^{N-1} \frac{1}{1-\Gamma-\nu}$$

$$+\psi\left[1+\Gamma\right]+\psi\left[1-\Gamma\right] .$$

So if we replace Γ by $\Gamma+N$, we get the same value for ℓ. This is not surprising because we know if γ is a solution $\gamma = N\ 2\pi/d$ is a solution. So we only need to compute the value of ℓ for $K<\Gamma<K+1$. But our preceding discussion has shown that our series is valid only for all square roots real; or

$$(n-\Gamma)^2-K^2>0 \quad , \text{ and because we study } \Gamma>0$$

$$n-\Gamma>K$$

$$\Gamma<n-K \quad .$$

We have supposed $K<1$ so Γ has only to satisfy

$\Gamma<1-K$ in the interval $K<\Gamma<K+1$.

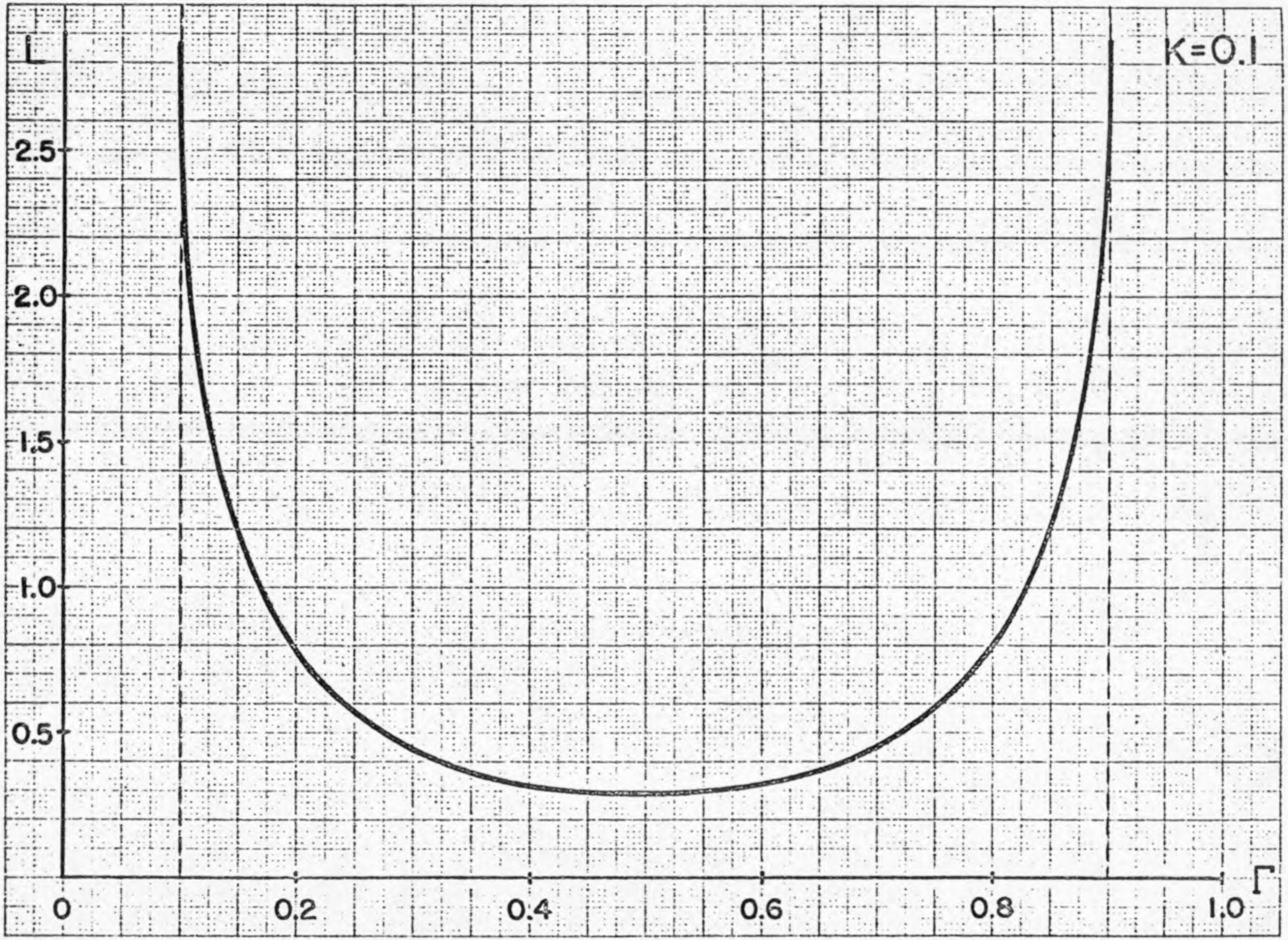
And finally

$$\boxed{K<\Gamma<1-K}$$

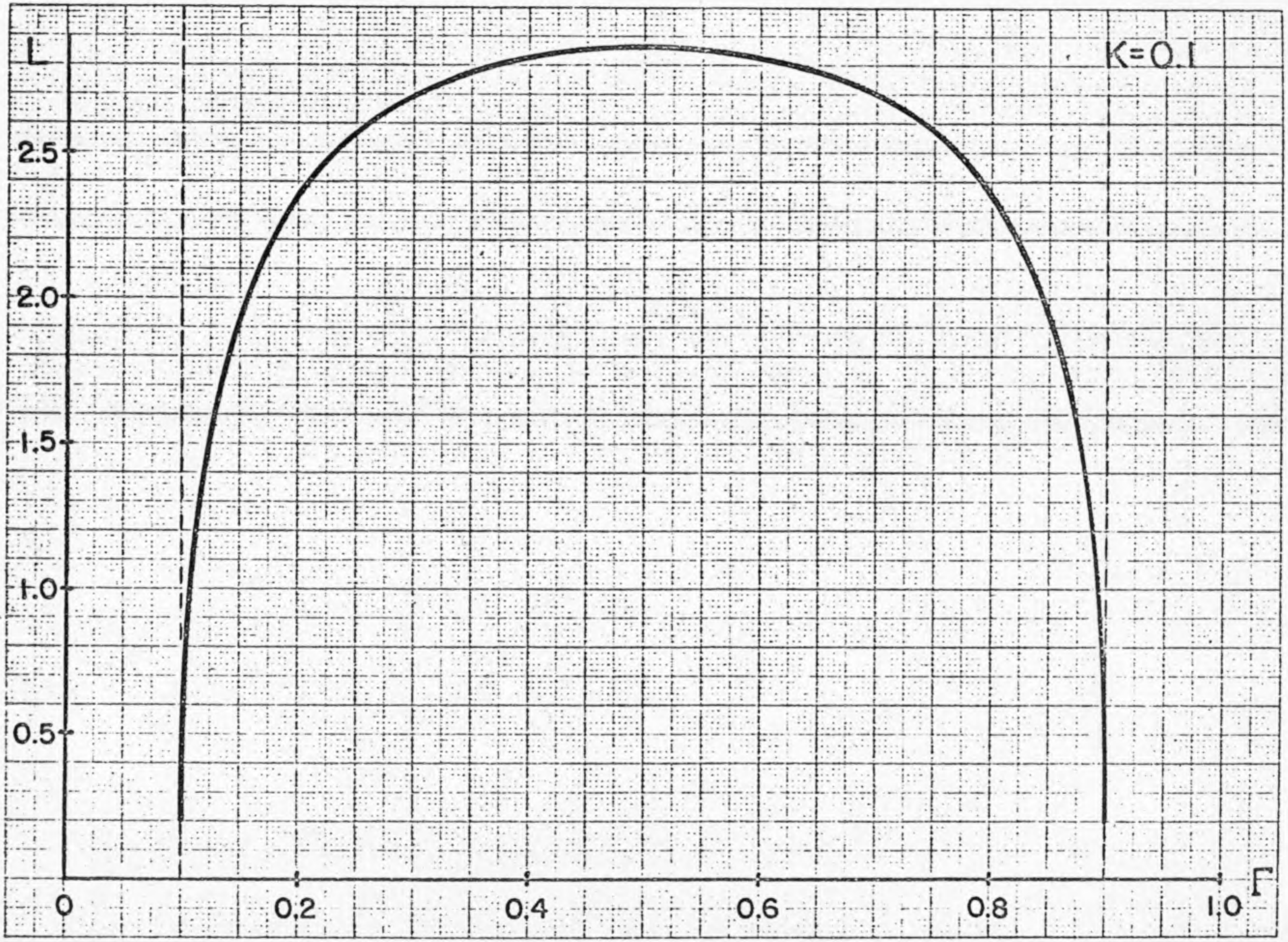
We have plotted (Fig.) ℓ as a function of γ in two cases:

1. $K = 1/10$ (10 strips by wavelength)
2. $K = 1/3$ (3 strips by wavelength)

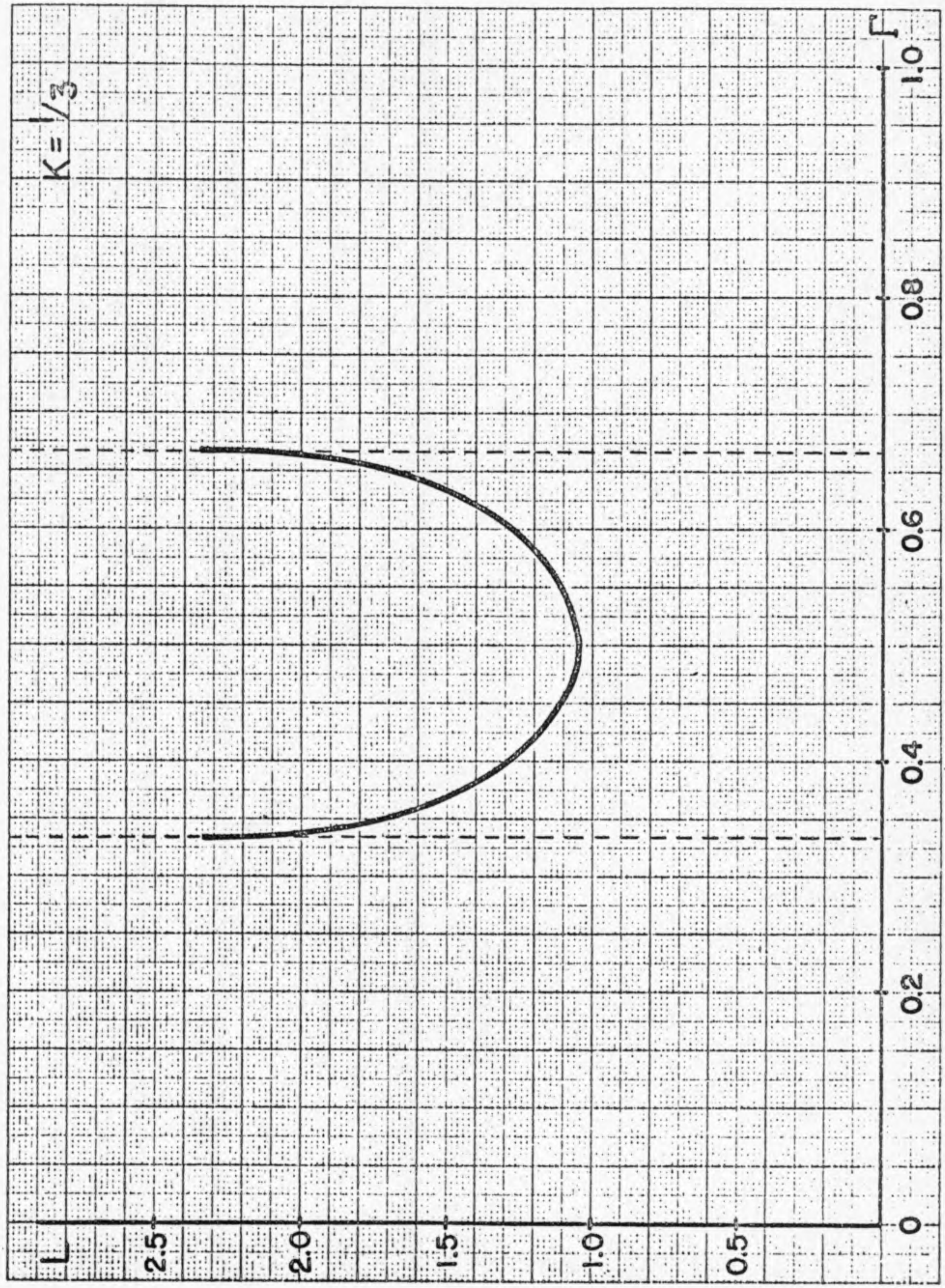
showing how d finite modifies the propagation constant. The curve is symmetric with respect to $\Gamma = 1/2$, for Γ and $-\Gamma+1$ give the same value for L. The approximation to the exact solution by this expression becomes worse when $L \to 0$.



ANTISYMMETRIC CASE



SYMMETRIC CASE



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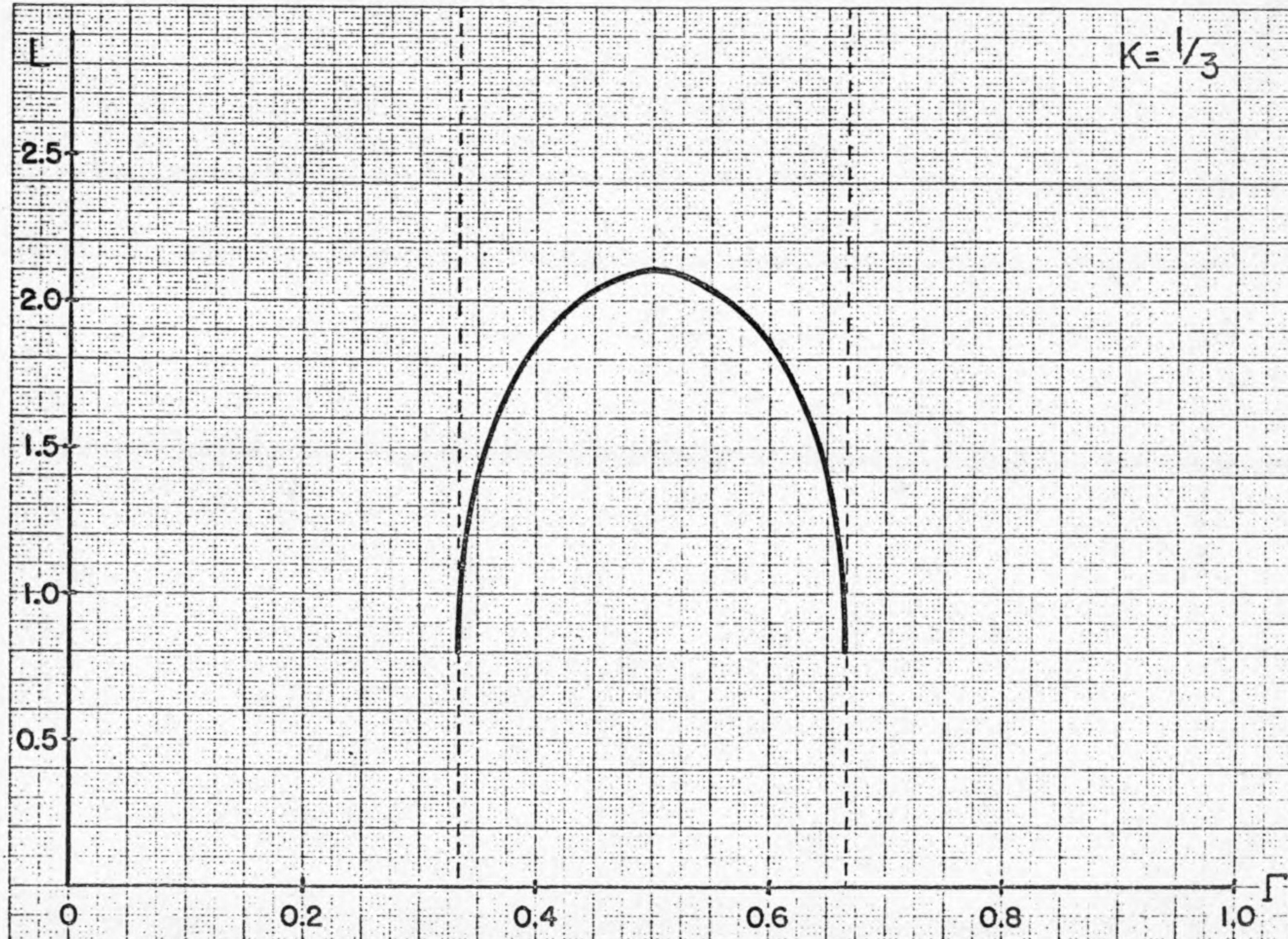

SYMMETRIC CASE

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